On definite integrals and infinite series*

Ernst Eduard Kummer

In the paper published in this journal, volume 12 p. 144, we gave theorems, by means of which infinite series, which contain the complete integral of the Riccati equation, can be expressed in terms of definite integrals, and there we also claimed that we will exhibit many other similar theorems. We will now present these theorems. For, since infinite series and definite integrals are one of the most simple and most frequently used forms, by which transcendental functions can be expressed, the transformation of one form into the other is to be considered of greatest importance. The transformation of definite integrals into series, by expansion of the function to be integrated, is very easy in most cases, but the other problem, on the transformation of infinite series into definite integrals, is a lot harder. For, in the solution of this problem a general method, which has a certain success, is missing, and only in a few cases peculiar artifices, or singular methods, lead to the propounded goal. Our theorems also contain certain special methods, and they can only be applied to certain classes of series, but they all have the same origin; and they are examples of a general method, which we will explain first.
For this aim, we recall the definite integral

$$\int\limits_a^b U \cdot f(u,k) du,$$

*Original Title: „De integralibus definitis et seriebus infinitis“, first published in Crelle Journal für reine und angewandte Mathematatik 17, 210-227 (1837); reprinted in „*Ernst Eduard Kummer Collected Papers II*, pp. 177 - 195“, translated by: Alexander Aycock for the project „Euler-Kreis Mainz“

in which U is a function of the variable u, $f(u,k)$ is a function of the quantities u and k, and k is an integer number, and let us assume that this integral satisfies this equation

$$1. \quad \int_a^b U \cdot f(u,k)du = B_k \int_a^b U \cdot f(u,0)du$$

in which B_k is a given function of the number k. Further, we assume the infinite series:

$$2. \quad A_0 f(u,0) + A_1 f(u,k) + A_2 f(u,2) + \cdots \text{etc.} = \phi(u)$$

whose sum $\phi(u)$ we assume to be expressible in terms of known functions. Having constituted these, it will be

$$\int_a^b U\phi(u)du$$

$$= A_0 \int_a^b Uf(u,0)du + A_1 \int_a^b Uf(u,1)du + A_2 \int_a^b Uf(u,2)du + \cdots \text{etc.}$$

and by formula (1.)

$$\int_a^b U\phi(u)du = \int_a^b Uf(u,0)[A_0B_0 + A_1B_1 + A_2B_2 + \cdots \text{etc.}]$$

or

$$3. \quad A_0B_0 + A_1B_1 + A_2B_2 + \cdots = \frac{\int_a^b U\phi(u)du}{\int_a^b Uf(u,0)du}.$$

By this method, if a certain series to be expressed in terms of definite integrals is propounded, the factors A_0, A_1, A_2 etc. are to separated from the single terms, and they are to be assumed in such a way that the sum of the series (2.) can be found easily; hence also the function $f(u,k)$ is to be chosen appropriately, after this it remains to find the function U satisfying equation (1.). Therefore, we would be able to express the sum of a certain series in terms of definite integrals in general, if the appropriate determination of the function U would succeed. But we discussed this problem, which is to be considered to be one of the more difficult ones, and requires singular methods for its solution, in another paper, but here will find certain forms of infinite series from several known integrals satisfying equation (1.), and we will deduce several theorems by means of the method just explained.

First, let us take the integral $\int_0^\infty e^{-u} \cdot u^{\alpha+k-1}du$, which Gauss denoted by $\Pi(\alpha + k - 1)$, which has the following known reduction:

$$4. \quad \int_0^\infty e^{-u}u^{\alpha+k-1}du = \alpha(\alpha+1)\cdots(\alpha+k-1)\int_0^\infty e^{-u}u^{\alpha-1}du,$$

which compared to equation (1.) gives $U = e^{-u}u^{\alpha-1}$, $F(u,k) = u^k$, $B_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, having substituted which value in the equations (2.) and (3.), we have.

Theorem I. "If the sum of the following series is known

$$5. \quad A_0 + A_1u + A_2u^2 + A_3u^3 + \cdots = \phi(u)$$

one has

$$6. \quad A_0 + \alpha A_1 + \alpha(\alpha+1)A_2 + \alpha(\alpha+1)(\alpha+2)A_3 + \cdots = \frac{\int_0^\infty e^{-u}u^{\alpha-1}\phi(u)du}{\int_0^\infty e^{-u}u^{\alpha-1}du}$$

or

$$7. \quad A_0 + \alpha A_1 + \alpha(\alpha+1)A_2 + \cdots + \text{etc.} = \frac{1}{\Pi(\alpha-1)}\int_0^\infty e^{-u}u^{\alpha-1}\phi(u)du."$$

We will explain the use of this theorem in some examples. If one puts

$$\phi(u) = \cos(u \cdot \tan x), \quad \text{then} \quad A_0 = 1, \quad A_1 = 0, \quad A_2 = -\frac{\tan^2 x}{1\cdot 2}, \quad A_3 = 0,$$

$$A_4 = +\frac{\tan^4 x}{1\cdot 2\cdot 3\cdot 4} \quad \text{etc.}$$

and hence by equation (7.)

$$1 - \frac{\alpha(\alpha+1)}{1\cdot 2}\tan^2 x + \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{1\cdot 2\cdot 3\cdot 4}\tan^4 x \cdots$$
$$= \frac{1}{\Pi(\alpha-1)}\int_0^\infty e^{-u}u^{\alpha-1}\cos(u\tan x)du,$$

substituting the known sum $(\cos x)^\alpha \cos(\alpha x)$ of this series, we have

$$8. \quad \int_0^\infty e^{-u}u^{\alpha-1}\cos(u\tan x)du = \Pi(\alpha-1)(\cos x)^\alpha\cos(\alpha x).$$

In like manner, if one takes $\phi(u) = \sin(u\tan x)$, whence $A_0 = 0$, $A_1 = \frac{\tan x}{1}$, $A_2 = 0$, $A_3 = -\frac{\tan^3 x}{1\cdot 2\cdot 3}$ etc., by equation (7.)

$$\frac{\alpha}{1}\tan x - \frac{\alpha(\alpha+1)(\alpha+2)}{1\cdot 2\cdot 3}\tan^3 x + \cdots = \frac{1}{\Pi(\alpha-1)}\int_0^\infty e^{-u}u^{\alpha-1}\sin(u\tan x)du,$$

and since the sum of this series is known, $(\cos x)^\alpha \sin \alpha x$, it follows:

$$9. \quad \int_0^\infty e^{-u}u^{\alpha-1}\sin(u\tan x)du = \Pi(\alpha-1)(\cos x)^\alpha\sin(\alpha x).$$

The two integrals, (8.) and (9.), we found by means of the theorem, were found by mathematicians a long time ago, and by various methods, but nevertheless the proof we gave seems to be better than others, since it does not require imaginary quantities, and holds for any arbitrary positive value, integer and fractional, of the number α. Generally, if the quantities A_0, A_1 etc. are assumed in such a way that not only $\phi(u)$, but also the sum of the series $A_0 + \alpha A_1 + \alpha(\alpha+1)A_2 + \cdots$ can be expressed in terms of known functions, the values of the definite integrals are found. Hence, if one puts $\phi(u) = \cos(2\sqrt{xu})$, we have $A_0 = 1$, $A_1 = -\frac{x}{\frac{1}{2}\cdot 1}$, $A_2 = \frac{x^2}{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2}$ etc. and hence

$$10. \quad 1 - \frac{\alpha\cdot x}{\frac{1}{2}\cdot 1} + \frac{\alpha(\alpha+1)x^3}{\frac{1}{2}\cdot\frac{1}{2}\cdot 1\cdot 2} - \cdots = \frac{1}{\Pi(\alpha-1)}\int_0^\infty e^{-u}u^{\alpha-1}\cos(2\sqrt{xu})du$$

furthermore, if one puts $\alpha = \frac{1}{2}$, this series goes over into the expansion of e^{-x}, and $\Pi(\alpha - 1) = \Pi(-\frac{1}{2}) = \sqrt{\pi}$, therefore,

$$\int_0^\infty e^{-u} u^{-\frac{1}{2}} \cos(2\sqrt{xu}) du = \sqrt{\pi} \cdot e^{-x},$$

or having put $u = v^2$ and $x = z^2$:

$$\int_0^\infty e^{-v^2} \cos(2zv) dv = \frac{\sqrt{\pi} \cdot e^{-z^2}}{2}.$$

We will deduce another theorem from the integral

$$\int_0^1 u^{\alpha-1} \left(\log \frac{1}{u}\right)^{\beta-1} du = \alpha^{-\beta} \Pi(\beta - 1)$$

which can be reduced by this formula:

$$11. \quad \int_0^1 u^{\alpha+k-1} \left(\log \frac{1}{u}\right)^{\beta-1} du = \frac{\alpha^\beta}{(\alpha+k)^\beta} \int_0^1 u^{\alpha-1} \left(\log \frac{1}{u}\right)^{\beta-1} du.$$

By comparison of this formula and equation (1.) for this case we have $f(u,k) = u^k$, $U = u^{\alpha-1} \left(\log \frac{1}{u}\right)^{\beta-1}$, $B_k = \left(\frac{\alpha}{\alpha+k}\right)^\beta$, therefore, from the equations (2.) and (3.) it follows:

Theorem II: "From the known sum of the series

$$A_0 + A_1 u + A_2 u^2 + \cdots = \phi(u)$$

it follows

$$14. \quad A_0 + \left(\frac{\alpha}{\alpha+1}\right)^\beta A_1 + \left(\frac{\alpha}{\alpha+2}\right)^\beta A_2 + \cdots = \frac{\int_0^1 u^{\alpha-1} \left(\log \frac{1}{u}\right)^{\beta-1} \phi(u) du}{\int_0^1 u^{\alpha-1} \left(\log \frac{1}{u}\right)^{\beta-1} du}$$

or

$$15.\quad \frac{A_0}{a^\beta} + \frac{A_1}{(a+1)^\beta} + \frac{A_2}{(a+2)^\beta} + \cdots = \frac{1}{\Pi(\beta-1)} \int_0^1 u^{\alpha-1} \left(\log\frac{1}{u}\right)^{\beta-1} \phi(u) du."$$

For the sake of an example, let us take

$$\phi(u) = \frac{x\cos\omega - x^2 u}{1 - 2xu\cos\omega + x^2u^2} = x\cos\omega + x^2 u\cos 2\omega + x^3u^2\cos 3\omega + \cdots$$

whence $A_0 = x\cos\omega$, $A_1 = x^2\cos 2\omega$, $A_3 = x^3\cos 3\omega$ etc.; therefore, by equation (15.)

$$16.\quad \frac{x\cos\omega}{\alpha^\beta} + \frac{x^2\cos 2\omega}{(\alpha+1)^\beta} + \frac{x^3\cos 3\omega}{(\alpha+2)^\beta} + \cdots$$

$$= \frac{1}{\Pi(\beta-1)} \int_0^1 \frac{u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1}(x\cos\omega - x^2u)du}{1 - 2xu\cos\omega + x^2u^2}.$$

The same integral, by means of which theorem II. was found, will give us another theorem by the following formula

$$17.\quad \int_0^1 u^{\alpha+k-1}\left(\log\frac{1}{u}\right)^{\beta+k-1} du = \frac{\beta(\beta+1)\cdots(\beta+k-1)\alpha^\beta}{(\alpha+k)^{\beta+k}} \int_0^1 u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1} du,$$

which, having compared it to equation (1.), gives:

$$f(u,k) = \left(u\log\frac{1}{u}\right)^k, \quad U = u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1}, \quad B_k = \frac{\beta(\beta+1)\cdots(\beta+k-1)\alpha^\beta}{(\alpha+k)^{\beta+k}}.$$

From these, having substituted them in eqs. (2.) and (3.), it follows:

Theorem III.: "By means of the known series:

$$18.\quad A_0 + A_1 u\log\frac{1}{u} + A_2\left(u\log\frac{1}{u}\right)^2 + \cdots = \phi\left(u\log\frac{1}{u}\right),$$

one also has the sum of this series expressed in terms of definite integrals:

$$19.\quad A_0+\frac{\beta\cdot\alpha^\beta}{(\alpha+1)^{\beta+1}}A_1+\frac{\beta\cdot(\beta+1)\alpha^\beta}{(\alpha+2)^{\beta+2}}A_2+\cdots=\frac{\int\limits_0^1 u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1}\phi\left(u\log\frac{1}{u}\right)du}{\int\limits_0^1 u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1}\phi du}$$

or

$$20.\quad A_0+\frac{\beta\cdot\alpha^\beta}{(\alpha+1)^{\beta+1}}A_1+\frac{\beta\cdot(\beta+1)\alpha^\beta}{(\alpha+2)^{\beta+2}}A_2+\cdots$$

$$\frac{1}{\Pi(\beta-1)}\int\limits_0^1 u^{\alpha-1}\left(\log\frac{1}{u}\right)^{\beta-1}\phi\left(u\log\frac{1}{u}\right)du.\text{”}$$

We can use this theorem to find the sum of the series $1+\frac{x}{2^2}+\frac{x^2}{3^3}+\frac{x^3}{4^4}+\text{etc.}$; for, if one takes $\alpha=1$, $\beta=1$ and $\phi\left(u\log\frac{1}{u}\right)=e^{xu\log\frac{1}{u}}=u^{-ux}$, we have $A_0=1$, $A_1=\frac{x}{1}$, $A_2=\frac{x^2}{1\cdot 2}$ etc., therefore, by equation (20.)

$$21.\quad 1+\frac{x}{2^2}+\frac{x^2}{3^3}+\frac{x^3}{4^4}+\text{etc.}=\int\limits_0^1 u^{-ux}du.$$

Here, we will also obtain that theorem, which we propounded already in this journal, volume 12, p. 144, and used to find several integrals. For this aim let us consider the integral

$$\int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1}du,$$

which is expressed in terms of the function Π this way

$$\int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1}\cdot du=\frac{\Pi(\alpha-1)\Pi(\beta-\alpha-1)}{\Pi(\beta-1)}$$

and one has this reduction formula:

$$22.\quad \int\limits_0^1 u^{\alpha+k-1}(1-u)^{\beta-\alpha-1}du=\frac{\alpha(\alpha+1)\cdots(\alpha+k-1))}{\beta(\beta+1)\cdots(\beta+k-1)}\int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1}du.$$

If this formula is compared to formula (1.), we have

Theorem IV: "From the known sum of the series

$$A_0 + A_1 + u + A_2u^2 + A_3u^3 + \cdots = \phi(u)$$

this sum of the series follows

$$24. \quad A_0 + \frac{\alpha}{\beta}A_1 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}A_2 + \cdots = \frac{\int_0^1 u^{\alpha+1}(1-u)^{\beta-\alpha-1}\phi(u)du}{\int_0^1 u^{\alpha+1}(1-u)^{\beta-\alpha-1}du}$$

or

$$25. \quad A_0 + \frac{\alpha}{\beta}A_1 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}A_2 + \cdots$$

$$= \frac{\Pi(\beta-1)}{\Pi(\alpha-1)\Pi(\beta-\alpha-1)} \int_0^1 u^{\alpha+1}(1-u)^{\beta-\alpha-1}\phi(u)du."$$

Putting $u = \sin^2 u$ we can represent equation (25.) in this form

$$26. \quad A_0 + \frac{\alpha}{\beta}A_1 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}A_2 + \cdots$$

$$= \frac{2\Pi(\beta-1)}{\Pi(\alpha)\Pi(\beta-\alpha-1)} \int_0^{\frac{\pi}{2}} (\sin v)^{2\alpha-1}(\cos v)^{2\beta-2\alpha-1}\phi(\sin^2 v)dv.$$

Now, if one puts $\phi(\sin^2 v) = \cos(2\beta v)$, from the known expansion

$$\cos(2\beta v) = 1 - \frac{\beta \cdot \beta}{\frac{1}{2} \cdot 1}\sin^2 v + \frac{\beta(\beta+1)\beta(\beta-1))}{\frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot 2}\sin^4 v - \cdots$$

it follows $A_0 = 1$, $A_1 = -\frac{\beta \cdot \beta}{\frac{1}{2} \cdot 1}$, $A_2 = +\frac{\beta(\beta+1)\beta(\beta-1)}{\frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot 2}$ etc., having substituted which in equation (26.), we find

$$27. \quad 1 - \frac{\alpha \cdot \beta}{\frac{1}{2} \cdot 1} + \frac{\alpha(\alpha+1)\beta(\beta-1)}{\frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot 2} - \cdots$$

$$= \frac{2\Pi(\beta-1)}{\Pi(\alpha)\Pi(\beta-\alpha-1)} \int\limits_0^{\frac{\pi}{2}} (\sin v)^{2\alpha-1}(\cos v)^{2\beta-2\alpha-1}\phi(\sin^2 v)dv,$$

but the sum of this series can be assigned in terms of the function Π (see Gauss Disquisit. gen. c. seriem inf. etc. p. 28)

$$1 - \frac{\alpha\cdot\beta}{\frac{1}{2}\cdot 1} + \frac{\alpha(\alpha+1)\beta(\beta-1)}{\frac{1}{2}\cdot\frac{1}{2}\cdot 1\cdot 2} - \cdots = \frac{\Pi\left(-\frac{1}{2}\right)\Pi\left(\beta-\alpha-\frac{1}{2}\right)}{\Pi\left(\beta-\frac{1}{2}\right)\Pi\left(-\alpha-\frac{1}{2}\right)},$$

hence equation (27.) goes over into this one:

$$\int\limits_0^{\frac{\pi}{2}} (\sin v)^{2\alpha-1}(\cos v)^{2\beta-2\alpha-1}\phi(\sin^2 v)dv = \frac{\Pi\left(-\frac{1}{2}\right)\Pi\left(\beta-\alpha-\frac{1}{2}\right)\Pi(\alpha-1)\Pi(\beta-\alpha-1)}{2\Pi(\beta-1)\Pi\left(\beta-\frac{1}{2}\right)\Pi\left(-\alpha-\frac{1}{2}\right)},$$

this expression is simplified tremendously by the fundamental formulas of the function Π, and obtains the simplest form, if α is changed into $\frac{\alpha}{2}$, β into $\frac{\alpha+\beta}{2}$, having done which, it results:

$$28. \quad \int\limits_0^{\frac{\pi}{2}} (\sin v)^{\alpha-1}(\cos v)^{\beta-1}\cos(\alpha+\beta)v dv = \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}.$$

In like manner, if in formula (26.) one sets

$$\phi(\sin^2 v) = \frac{\sin(2\beta-1)v}{(2\beta-1)\sin v},$$

one will find the integral

$$29. \quad \int\limits_0^{\frac{\pi}{2}} (\sin v)^{\alpha-1}(\cos v)^{\beta-1}\sin(\alpha+\beta)v dv = \frac{\sin\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}.$$

But this integral is easily deduced from that one; for, if in that one v is changed into $\frac{\pi}{2}-v$, α into β and β into α, it results

$$\cos\left((\alpha+\beta)\frac{\pi}{2}\right)\int\limits_0^{\frac{\pi}{2}} (\sin v)^{\alpha-1}(\cos v)^{\beta-1}\cos(\alpha+\beta)v dv$$

$$+\sin\left((\alpha+\beta)\frac{\pi}{2}\right)\int\limits_0^{\frac{\pi}{2}}(\sin v)^{\alpha-1}(\cos v)^{\beta-1}\sin(\alpha+\beta)v\,dv=\frac{\cos\frac{\beta\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}$$

from which by formula (28.) is follows

$$\int\limits_0^{\frac{\pi}{2}}(\sin v)^{\alpha-1}(\cos v)^{\beta-1}\sin(\alpha+\beta)v\,dv$$

$$=\frac{\left(\cos\frac{\beta\pi}{2}-\cos\frac{\alpha\pi}{2}\cos\left((\alpha+\beta)\frac{\pi}{2}\right)\right)\Pi(\alpha-1)\Pi(\beta-1)}{\sin\left((\alpha+\beta)\frac{\pi}{2}\right)\cdot\Pi(\alpha+\beta-1)},$$

which formula is identical to equation (29.), since

$$\frac{\cos\frac{\beta\pi}{2}-\cos\frac{\alpha\pi}{2}\cos(\alpha+\beta)\frac{\pi}{2}}{\sin(\alpha+\beta)\frac{\pi}{2}}=\sin\frac{\alpha\pi}{2}.$$

Another integral is found by means of the fourth theorem by putting $\alpha=\frac{1}{2}$, $\phi(\sin^2 v)=\cos(\gamma v)$, hence by the known expansion

$$\cos(\gamma v)=1-\frac{\frac{\gamma}{2}\cdot\frac{\gamma}{2}}{\frac{1}{2}\cdot 1}\sin^2 v+\frac{\frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)}{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2},\quad\text{etc.}$$

having substituted which, it results

$$1-\frac{\frac{\gamma}{2}\cdot\frac{\gamma}{2}}{\beta\cdot 1}+\frac{\frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)}{\beta(\beta+1)1\cdot 2}-\cdots$$

$$=\frac{2\Pi(\beta-1)}{\Pi\left(-\frac{1}{2}\right)\Pi\left(\beta-\frac{1}{2}\right)}\int\limits_0^{\frac{\pi}{2}}(\cos v)^{2\beta-2}\cos(\gamma v)\,dv,$$

and since the sum of this series can be expressed in terms of the function Π this way:

$$\frac{\Pi(\beta-1)\Pi(\beta-1)}{\Pi\left(\beta+\frac{\gamma}{2}-1\right)\Pi\left(\beta-\frac{\gamma}{2}-1\right)}$$

we have

$$\int\limits_0^{\frac{\pi}{2}}(\cos v)^{2\beta-2}\cos(\gamma v)\,dv=\frac{\Pi\left(-\frac{1}{2}\right)\Pi(\beta-1)\Pi\left(\beta-\frac{1}{2}\right)}{2\Pi\left(\beta+\frac{\gamma}{2}-1\right)\Pi\left(\beta-\frac{\gamma}{2}-1\right)},$$

finally, if β is changed into $\frac{\beta+2}{2}$, and $\Pi\left(\frac{\beta}{2}\right)\Pi\left(\frac{\beta-1}{2}\right)$ is transformed into $\sqrt{\pi}\cdot 2^{-\beta}\Pi(\beta)$, this integral will take on the form:

$$30. \quad \int_0^{\frac{\pi}{2}} (\cos v)^\beta \cos(\gamma v) = \frac{\pi\Pi(\beta)}{2^{\beta+1}\Pi\left(\frac{\beta+\gamma}{2}\right)\Pi\left(\frac{\beta-\gamma}{2}\right)}.$$

By the same method one can find the two integrals

$$\int_0^{\pi} (\sin v)^\beta \cos(\gamma v) dv \quad \text{and} \quad \int_0^{\pi} (\sin v)^\beta \sin(\gamma v) dv$$

but they are deduced more easily from the one we just found, which for this purpose can be represented this way

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos v)^\beta \cos(\gamma v) dv = \frac{\pi\Pi(\beta)}{2^{\beta}\Pi\left(\frac{\beta+\gamma}{2}\right)\Pi\left(\frac{\beta-\gamma}{2}\right)}$$

and from this

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos v)^\beta \sin(\gamma v) dv = 0,$$

whose validity is obvious. For, if that one is multiplied by $\cos\frac{\gamma\pi}{2}$, this one by $\sin\frac{\gamma\pi}{2}$, by addition:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos v)^\beta \cos\left(\gamma v - \frac{\gamma\pi}{2}\right) dv = \frac{\pi \cos\frac{\gamma\pi}{2}\Pi(\beta)}{2^{\beta}\Pi\left(\frac{\beta+\gamma}{2}\right)\Pi\left(\frac{\beta-\gamma}{2}\right)},$$

finally, if v is changed into $\frac{\pi}{2} - v$, we have

$$31. \quad \int_0^{\pi} (\sin v)^\beta \cos(\gamma v) dv = \frac{\pi \cos\left(\frac{\gamma\pi}{2}\right)\Pi(\beta)}{2^{\beta}\Pi\left(\frac{\beta+\gamma}{2}\right)\Pi\left(\frac{\beta-\gamma}{2}\right)},$$

$$32. \quad \int_0^{\pi} (\sin v)^\beta \sin(\gamma v) dv = \frac{\pi \sin\left(\frac{\gamma\pi}{2}\right)\Pi(\beta)}{2^{\beta}\Pi\left(\frac{\beta+\gamma}{2}\right)\Pi\left(\frac{\beta-\gamma}{2}\right)}.$$

From the found value of the integral (30.) or in other ways only easily deduces the reduction formula:

$$\int_0^{\frac{\pi}{2}} (\cos u)^{\beta+2k-1} \cos(\gamma u) du = B_k \int_0^{\frac{\pi}{2}} (\cos u)^{\beta-1} \cos(\gamma u) du,$$

in which B_k denotes this expression:

$$B_k = \frac{\beta(\beta+1)(\beta+2)\cdots(\beta+2k-1))}{(\beta+\gamma+1)(\beta+\gamma+3)\cdots(\beta+\gamma+2k-1)(\beta-\gamma+1)(\beta-\gamma+3)\cdots(\beta-\gamma+2k-1)},$$

this formula yields the theorem:

Theorem V: "If one puts

$$A_0 + A_1 \cos^2 u + A_2 \cos^4 u + A_3 \cos^6 u + \cdots = \phi(\cos^2 u)$$

and

$$34. \quad R = A_0 + \frac{\beta(\beta+1)A_1}{(\beta+\gamma+1)(\beta-\gamma+1)} + \frac{\beta(\beta+1)(\beta+2)(\beta+3)A_2}{(\beta+\gamma+1)(\beta+\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)} + \cdots$$

we find

$$35. \quad R = \frac{\int_0^{\frac{\pi}{2}} (\cos u)^{\beta-1} \cos(\gamma u) \phi(\cos^2 u) du}{\int_0^{\frac{\pi}{2}} (\cos u)^{\beta-1} \cos(\gamma u) du}$$

or

$$36. \quad R = \frac{2^\beta \Pi\left(\frac{\beta+\gamma-1}{2}\right) \Pi\left(\frac{\beta-\gamma-1}{2}\right)}{\pi \Pi(\beta-1)} \int_0^{\frac{\pi}{2}} (\cos u)^{\beta-1} \cos(\gamma u) \phi(\cos^2 u) du."$$

We have a simple example of this theorem for $\beta = 1$, $\gamma = 2\alpha$, $\phi(\cos^2 u) = \cos(2z\cos u)$, whence $A_0 = 1$, $A_1 = -\frac{2^2 z^2}{1\cdot 2}$, $A_2 = +\frac{2^4 z^4}{1\cdot 2\cdot 3\cdot 4}$ etc., having substituted which values in equation (36.):

$$37. \quad 1 - \frac{z^2}{(1+\alpha)(1-\alpha)} + \frac{z^4}{(1+\alpha)(2+\alpha)(1-\alpha)(2-\alpha)} - \cdots$$
$$= \frac{2\alpha}{\sin\alpha\pi}\int_0^{\frac{\pi}{2}} \cos(2\alpha u)\cos(2z\cos u)du.$$

Another theorem is deduced from the formula

$$\int_0^{\frac{\pi}{2}} (\cos v)^{\beta-\alpha-1}\cos(\beta+\alpha+2k-1)v dv = B_k \int_0^{\frac{\pi}{2}} (\cos v)^{\beta-\alpha-1}\cos(\beta+\alpha-1)v dv,$$

where

$$B_k = (-1)^k \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta(\beta+1)\cdots(\beta+k-1)},$$

which is easily demonstrated applying equation (30.) or by other methods. By comparison of this formula to equation (1.) we see that in this case one has to set

$$f(u,k) = \cos(\beta+\alpha+2k-1)u, \quad U = (\cos u)^{\beta-\alpha-1},$$

which values substituted in the equations (2.) and (3.) give:

Theorem VI. "If the sum of the following series is known

$$38. \quad A_0\cos(\alpha+\beta-1)u + A_1\cos(\alpha+\beta+1)u + A_2\cos(\alpha+\beta+3)u + \cdots = \phi(u)$$

one also has the sum of this series

$$39. \quad A_0 - \frac{\alpha}{\beta}A_1 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}A_2 - \cdots = \frac{\int_0^{\frac{\pi}{2}} (\cos u)^{\beta-\alpha-1}\phi(u)du}{\int_0^{\frac{\pi}{2}} (\cos u)^{\beta-\alpha-1}du}$$

or

$$40.\quad A_0 - \frac{\alpha}{\beta}A_1 + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}A_2 - \cdots = \frac{2^{\beta-1}\Pi(\beta-1)\Pi(-\alpha)}{\pi\Pi(\beta-\alpha-1)}\int\limits_0^{\frac{\pi}{2}}(\cos u)^{\beta-\alpha-1}\phi(u)du.\text{''}$$

If we use this theorem to find the sum of the series

$$1 + \frac{x}{\beta\cdot 1} + \frac{x^2}{\beta(\beta+1)\cdot 1\cdot 2} + \cdots,$$

it has to be $\alpha = \frac{1}{2}$ and $A_0 = 1$, $A_1 = \frac{x}{\frac{1}{2}\cdot 1}$, $A_2 = +\frac{x^2}{\frac{1}{2}\cdot\frac{1}{2}\cdot 1\cdot 2}$ etc., one has to find the sum of this series

$$\phi(u) = \cos\left(\beta - \frac{1}{2}\right)u - \frac{x\cos\left(\beta+\frac{1}{2}\right)u}{\frac{3}{2}\cdot 1} + \frac{x^2\cos\left(\beta+\frac{7}{2}\right)}{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2} - \cdots$$

which by known methods is derived from

$$\cos(2\sqrt{x}) = 1 - \frac{x}{\frac{1}{2}\cdot 1} + \frac{x^2}{\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot 2} - \cdots;$$

of course

$$\begin{aligned}\phi(u) = {} & \frac{1}{2}e^{2\sqrt{x}\sin u}\cos\left(\left(\beta-\frac{1}{2}\right)u - 2\sqrt{x}\cos u\right) \\ & + \frac{1}{2}e^{-2\sqrt{x}\sin u}\cos\left(\left(\beta-\frac{1}{2}\right)u + 2\sqrt{x}\cos u\right)\end{aligned}$$

having substituted which in equation (40.) we have

$$1 + \frac{x}{\beta\cdot 1} + \frac{x^2}{\beta(\beta+1)\cdot 1\cdot 2} + \cdots = \frac{2^{\beta-\frac{1}{2}}\Pi(\beta-1)}{\sqrt{\pi}\Pi\left(\beta-\frac{1}{2}\right)}\int\limits_0^{\frac{\pi}{2}}(\cos u)^{\beta-\frac{1}{2}}\phi(u)du.$$

The function $\phi(u)$ consists of two parts, which only differ by the sign of the quantity $\sqrt{x}$ and it is easily clear, if we integrate each of both parts separately and write $-v$ instead of v in one of them, that these two integrals differ only in regard to their limits, which are $-\frac{\pi}{2}$ and $= 0$ for the one, 0 and $+\frac{\pi}{2}$ for the other. Therefore, having combined these integrals, we have

$$41.\quad 1+\frac{x}{\beta\cdot 1}+\frac{x^2}{\beta(\beta+1)\cdot 1\cdot 2}+\cdots$$

$$=\frac{2^{\beta-\frac{1}{2}}\Pi(\beta-1)}{\sqrt{\pi}\Pi\left(\beta-\frac{1}{2}\right)}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}(\cos u)^{\beta-\frac{1}{2}}e^{2\sqrt{x}\sin u}\cos\left(\left(\beta-\frac{1}{2}\right)u-2\sqrt{x}\cos u\right)du.$$

Let us return to the integral found above of equation (11.)

$$\int\limits_0^\infty e^{-v^2}\cos(2zv)dv=\frac{\sqrt{\pi}}{2}e^{-z^2}$$

which, having put $z=\alpha\sqrt{\log\frac{1}{q}}$, is transformed into

$$\int\limits_0^\infty e^{-v^2}\cos\left(2\alpha v\sqrt{\log\frac{1}{q}}\right)dq=\frac{\sqrt{\pi}}{2}q^{\alpha^2},$$

hence, if one writes $\alpha+k$ instead of α, one deduces this formula

$$42.\quad \int\limits_0^\infty e^{-v^2}\cos\left(2(\alpha+k)v\sqrt{\log\frac{1}{q}}\right)dv$$

$$=q^{k^2+2\alpha k}\int\limits_0^\infty e^{-v^2}\cos\left(2\alpha v\sqrt{\log\frac{1}{q}}\right)dv.$$

This formula will give us a new most remarkable theorem. For, by comparison with equation (1.) we have $f(u,k)=\cos\left(2(\alpha+k)v\sqrt{\log\frac{1}{q}}\right)$, $U=e^{-v^2}$, $B_k=q^{k^2+2\alpha k}$; therefore, from the equations (2.) and (3.) it follows:

Theorem VII. "If the sum of the following series is known

$$43.\quad A_0\cos\left(2\alpha v\sqrt{\log\tfrac{1}{q}}\right)+A_1\cos\left(2(\alpha+1)v\sqrt{\log\tfrac{1}{q}}\right)$$

$$+A_2\cos\left(2(\alpha+2)v\sqrt{\log\tfrac{1}{q}}\right)+\cdots=\phi(v)$$

one also has the sum of this series

$$44. \quad A_0 + A_1 q^{1+2\alpha} + A_2 q^{4+4\alpha} + A_3 q^{9+6\alpha} + \cdots = \frac{\int\limits_0^\infty e^{-v^2} \phi(v) dv}{\int\limits_0^\infty e^{-v^2} \cos\left(2\alpha v \sqrt{\log \frac{1}{q}}\right) dv}$$

or

$$45. \quad A_0 + A_1 q^{1+2\alpha} + A_2 q^{4+4\alpha} + A_3 q^{9+6\alpha} + \cdots = \frac{2q^{-\alpha^2}}{\sqrt{\pi}} \int\limits_0^\infty e^{-v^2} \phi(v) dv.''$$

By means of this theorem one can find the sums of series containing powers of the quantity q, whose exponents proceed in an arithmetic progression of *second* order. On finds some very simple series in the theory of elliptic functions, to which we will mainly apply our method here. For this aim let us take

$$\phi(v) = \cos\left(2\beta v \sqrt{\log \frac{1}{q}}\right) + x \cos\left(2(\beta - 1) v \sqrt{\log \frac{1}{q}}\right) + \cos\left(2(\beta - 2) v \sqrt{\log \frac{1}{q}}\right) + \cdots$$

the sum of which series is

$$\phi(v) = \frac{\cos\left(2\beta v \sqrt{\log \frac{1}{q}}\right) - x \cos\left(2(\beta + 1) v \sqrt{\log \frac{1}{q}}\right)}{1 - 2x \cos\left(2v \sqrt{\log \frac{1}{q}}\right) + x^2}$$

whence $A_0 = 1$, $A_1 = x$, $A_2 = x^2$ etc., $\alpha = \beta$, having substituted the values in equation (45.), we have

$$46. \quad 1 + xq^{1-2\beta} + x^2 q^{4-4\beta} + x^3 q^{9-6\beta} + \cdots$$

$$= \frac{2q^{-\beta^2}}{\sqrt{\pi}} \int\limits_0^\infty \frac{e^{-v^2} \left[\cos\left(2\beta v \sqrt{\log \frac{1}{q}}\right) - x \cos\left(2(\beta + 1) v \sqrt{\log \frac{1}{q}}\right)\right]}{1 - 2x \cos\left(2v \sqrt{\log \frac{1}{q}}\right) + x^2} dv$$

and, if one puts $x = zq^{2\beta}$,

$$47.\quad 1+zq+z^2q^4+z^3q^9+z^4q^{16}+\cdots$$
$$=\frac{2q^{-\beta^2}}{\sqrt{\pi}}\int\limits_0^\infty \frac{e^{-v^2}\left[\cos\left(2\beta v\sqrt{\log\frac{1}{q}}\right)-zq^{2\beta}\cos\left(2(\beta+1)v\sqrt{\log\frac{1}{q}}\right)\right]}{1-2zq^{2\beta}\cos\left(2v\sqrt{\log\frac{1}{q}}\right)+z^2q^{4\beta}}dv.$$

The quantity β, which is not contained in the one side of the equation, can be chosen arbitrarily, but nevertheless one has to note, that it is taken in such a way that $x = zq^{2\beta}$ is smaller than 1 for $\phi(v)$ not to become a divergent series. The integral would have the simplest form for $\beta = 0$; but then it would not be possible to attribute the value 1 to the quantity z. For this reason let us take $\beta = \frac{1}{2}$, and furthermore, let us set $z = +1$ and $z = -1$; hence we will obtain these series

$$48.\quad 1+q+q^4+q^9+q^{16}+\cdots$$
$$=\frac{2}{\sqrt{\pi}\sqrt[4]{q}}\int\limits_0^\infty \frac{e^{-v^2}\left[\cos\left(v\sqrt{\log\frac{1}{q}}\right)-q\cos\left(3v\sqrt{\log\frac{1}{q}}\right)\right]}{1-2q\cos\left(2v\sqrt{\log\frac{1}{q}}\right)+q^2}dv,$$
$$49.\quad 1-q+q^4-q^9+q^{16}-\cdots$$
$$=\frac{2}{\sqrt{\pi}\sqrt[4]{q}}\int\limits_0^\infty \frac{e^{-v^2}\left[\cos\left(v\sqrt{\log\frac{1}{q}}\right)+q\cos\left(3v\sqrt{\log\frac{1}{q}}\right)\right]}{1+2q\cos\left(2v\sqrt{\log\frac{1}{q}}\right)+q^2}dv.$$

Jacobi (Fund. nova theor. f. ell. p. 184) found the same two series

$$1+q+q^4+q^9+q^{16}+\cdots=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2K}{\pi}},$$
$$1-q+q^4-q^9+q^{16}-\cdots=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2k'K}{\pi}},$$

where the quantities q, K and k' depend on the variable k in such a way that

$$k'=\sqrt{1-k^2},\quad K=\int\limits_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},\quad K'=\int\limits_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}},$$
$$q=e^{-\frac{\pi K'}{K}}.$$

In like manner in theorem VII. let us take

$$\phi(v) = \cos\left(2(\beta-1)v\sqrt{\log\tfrac{1}{q}}\right) + x\cos\left(2(\beta-3)v\sqrt{\log\tfrac{1}{q}}\right)$$
$$+ x^3\cos\left(2(\beta-5)v\sqrt{\log\tfrac{1}{q}}\right) + \cdots$$

whence it follows $\alpha = -\beta$ and $A_0 = 0$, $A_1 = 1$, $A_2 = 0$, $A_3 = x$, $A_4 = 0$, $A_5 = x^3$ etc. The sum of the series $\phi(v)$ is easily found

$$\phi(v) = \frac{\cos\left(2(\beta-1)v\sqrt{\log\tfrac{1}{q}}\right) - x\cos\left(2(\beta+1)v\sqrt{\log\tfrac{1}{q}}\right)}{1-2x\cos\left(4v\sqrt{\log\tfrac{1}{q}}\right)+x^2}$$

hence by equation (45.) we have

$$50.\quad q^{1-2\beta} + xq^{9-6\beta} + x^3q^{25-10\beta} + x^3q^{49-14\beta} + \cdots$$
$$= \frac{2q^{-\beta^2}}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}} \frac{e^{-v^2}\left[\cos\left(2(\beta-1)v\sqrt{\log\tfrac{1}{q}}\right) - x\cos\left(2(\beta+1)v\sqrt{\log\tfrac{1}{q}}\right)\right]}{1-2x\cos\left(4v\sqrt{\log\tfrac{1}{q}}\right)+x^2}dv$$

putting $x = zq^{4\beta}$ and $\beta = 1$ and multiplying by q^2, we have

$$51.\quad q + zq^9 + z^2q^{25} + z^3q^{49} + \cdots$$
$$= \frac{2q}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}} \frac{e^{-v^2}\left[1 - zq^4\cos\left(4v\sqrt{\log\tfrac{1}{q}}\right)\right]}{1-2zq^4\cos\left(4v\sqrt{\log\tfrac{1}{q}}\right)+z^2q^8}dv,$$

if q is changed into $\sqrt[4]{q}$, this formula goes over into this one

$$52.\quad \sqrt[4]{q} + z\sqrt[4]{q^9} + z^2\sqrt[4]{q^{25}} + z^3\sqrt[4]{q^{49}} + \cdots$$
$$= \frac{2\sqrt[4]{q}}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}} \frac{e^{-v^2}\left[1 - zq\cos\left(4v\sqrt{\log\tfrac{1}{q}}\right)\right]}{1-2zq\cos\left(2v\sqrt{\log\tfrac{1}{q}}\right)+z^2q^2}dv,$$

finally, putting $z = 1$ and $z = -1$, we have these two series

$$53. \quad \sqrt[4]{q} + \sqrt[4]{q^9} + \sqrt[4]{q^{25}} + \sqrt[4]{q^{49}} + \cdots$$
$$= \frac{2\sqrt[4]{q}}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \frac{e^{-v^2}\left[1 - q\cos\left(4v\sqrt{\log\frac{1}{q}}\right)\right]}{1 - 2q\cos\left(2v\sqrt{\log\frac{1}{q}}\right) + q^2} dv,$$
$$54. \quad \sqrt[4]{q} - \sqrt[4]{q^9} + \sqrt[4]{q^{25}} - \sqrt[4]{q^{49}} + \cdots$$
$$= \frac{2\sqrt[4]{q}}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \frac{e^{-v^2}\left[1 + q\cos\left(4v\sqrt{\log\frac{1}{q}}\right)\right]}{1 + 2q\cos\left(2v\sqrt{\log\frac{1}{q}}\right) + q^2} dv,$$

of which the first was found by Jacobi in his book mentioned above

$$\sqrt[4]{q} + \sqrt[4]{q^9} + \sqrt[4]{q^{25}} + \sqrt[4]{q^{49}} + \cdots = \sqrt{\frac{kK}{2\pi}}.$$

By the same method one can also find the sums of the more general series

$$\theta\left(\frac{2Kx}{\pi}\right) = 1 - 2q\cos x + 2q^4\cos 4x - 2q^9\cos 6x + \cdots,$$
$$H\left(\frac{2Kx}{\pi}\right) = 2\sqrt[4]{q}\cdot\sin x - 2\sqrt[4]{q^9}\cdot\sin 3x + 2\sqrt[4]{q^{25}}\cdot\sin 5x - \cdots,$$

which occur frequently in the theory of elliptic functions, whose expressions in terms of definite integrals, since they become less simple, we omit here.

From the known integral

$$\int_0^1 \frac{u^{\alpha-1} - u^{\beta-1}}{\log u} du = \log\left(\frac{\alpha}{\beta}\right)$$

it immediately follows:
Theorem VIII. "If one puts

$$A_0 + A_1 u + A_2 u^2 + A_3 u^3 + \cdots = \phi(u)$$

we have

$$57.\quad A_0 \log\left(\frac{\alpha}{\beta}\right) + A_1 \log\left(\frac{\alpha+1}{\beta+1}\right) + A_2 \log\left(\frac{\alpha+2}{\beta+2}\right) + \cdots = \int_0^1 \frac{u^{\alpha-1} - u^{\beta-1}}{\log u} \phi(u) du.''$$

We obtain a remarkable example of this theorem by putting

$$\phi(u) = 1 - u + u^2 - u^3 + \cdots + u^{2n} = \frac{1 + u^{2n+1}}{1+u},$$

whence:

$$58.\quad \log\left(\frac{\alpha}{\beta}\right) - \log\left(\frac{\alpha+1}{\beta+1}\right) + \log\left(\frac{\alpha+2}{\beta+2}\right) - \cdots + \log\left(\frac{\alpha+2n}{\beta+2n}\right) = \int_0^1 \frac{(u^{\alpha-1} - u^{\beta-1})(1+u^{2n+1})}{(1+u)\log u} du,$$

this sum of logarithms is collected into one logarithm of this product

$$\frac{\alpha(\alpha+2)\cdots(\alpha+2n)(\beta+1)(\beta+3)\cdots(\beta+2n-1)}{\beta(\beta+2)\cdots(\beta+2n)(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}$$

and, if we, following Gauss, put

$$\Pi(k,z) = \frac{1 \cdot 2 \cdot 3 \cdots k \cdot k^z}{(z+1)(z+2)\cdots(z+k)}$$

this product can be represented this way

$$\left(\frac{n}{n+1}\right)^{\frac{\beta-\alpha}{2}} \cdot \frac{\Pi\left(n+1, \frac{\beta}{2}-1\right)\Pi\left(n, \frac{\alpha-1}{2}\right)}{\Pi\left(n+1, \frac{\alpha}{2}-1\right)\Pi\left(n, \frac{\beta-1}{2}\right)},$$

therefore,

$$59.\quad \int_0^1 \frac{(u^{\alpha-1} - u^{\beta-1})(1+u^{2n+1})}{(1+u)\log u} du = \log\left\{\left(\frac{n}{n+1}\right)^{\frac{\beta-\alpha}{2}} \cdot \frac{\Pi\left(n+1, \frac{\beta}{2}-1\right)\Pi\left(n, \frac{\alpha-1}{2}\right)}{\Pi\left(n+1, \frac{\alpha}{2}-1\right)\Pi\left(n, \frac{\beta-1}{2}\right)}\right\},$$

if the number n becomes infinitely large, u^{2n+1} in the integral vanishes and $\Pi(n,z)$ goes over into $\Pi(z)$, whence:

$$60.\quad \int_0^1 \frac{(u^{\alpha-1} - u^{\beta-1})}{(1+u)\log u} du = \log\left\{\frac{\Pi\left(\frac{\beta}{2}-1\right)\Pi\left(\frac{\alpha-1}{2}\right)}{\Pi\left(\frac{\alpha}{2}-1\right)\Pi\left(\frac{\beta-1}{2}\right)}\right\}.$$

From this integral, which we believe to be new, also this special case follows

$$61. \quad \int_0^1 \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log u} = \log\left(\tan\frac{\alpha\pi}{2}\right).$$

Many other theorems can be deduced from other values and reductions of definite integrals applying the general method mentioned above; but to collect them all, would take a lot of time. And those, we gave here, shall suffice as examples of the general method. But it is convenient to add a theorem of another kind here, which is connected to the others and is demonstrated by applying them.

Theorem IX. "If a certain function has an expansion of this form

$$62. \quad \phi(\cos^2 u) = A_0 + A_1\cos^2 u + A_2\cos^4 u + A_3\cos^6 u + \cdots$$

and the same function is expanded into a series of this form:

$$63. \quad \phi(\cos^2 u) = B_0 + \frac{B_1\cos u}{2\cos u} + \frac{B_2\cos 2u}{(2\cos u)^2} + \frac{B_3\cos 3u}{(2\cos u)^3} + \cdots$$

the coefficients B_0, B_1, B_2 etc. are determined by definite integrals in such a way that in general

$$64. \quad B_h = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} (2\cos u)^{k-1}\cos(h+1)u\phi(\cos^2 u)du."$$

The proof of this theorem is based on the formula

$$65. \quad (\cos u)^{2k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\left(1 + \frac{2k}{k+1}\cdot\frac{\cos u}{2\cos u} + \frac{2k(2k+1)}{(k+1)(k+2)}\cdot\frac{\cos 2u}{(2\cos u)^2} + \cdots\right),$$

which is easily deduced from a more general formula, I propounded in this journal (Volume 15 p. 161 form. 13). For, if in the equation

$$\phi(\cos^2 u) = A_0 + A_1\cos^2 u + A_2\cos^4 u + \cdots$$

one substitutes the expressions, which equation (65.) yields, for the powers of the cosines, we find

65. $\phi(\cos^2 u) = A_0$

$$+\frac{1}{2}A_1\left(1+\frac{2}{2}\cdot\frac{\cos u}{2\cos u}+\frac{2\cdot 3}{2\cdot 3}\cdot\frac{\cos 2u}{(2\cos u)^2}+\frac{2\cdot 3\cdot 4}{2\cdot 3\cdot 4}\cdot\frac{\cos 3u}{(2\cos u)^3}+\cdots\right)$$
$$+\frac{1\cdot 3}{2\cdot 4}A_2\left(1+\frac{4}{3}\cdot\frac{\cos u}{2\cos u}+\frac{4\cdot 5}{3\cdot 4}\cdot\frac{\cos 2u}{(2\cos u)^2}+\frac{4\cdot 5\cdot 6}{3\cdot 4\cdot 5}\cdot\frac{\cos 3u}{(2\cos u)^3}+\cdots\right)$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}A_3\left(1+\frac{6}{4}\cdot\frac{\cos u}{2\cos u}+\frac{6\cdot 7}{4\cdot 5}\cdot\frac{\cos 2u}{(2\cos u)^2}+\frac{6\cdot 7\cdot 8}{4\cdot 5\cdot 6}\cdot\frac{\cos 3u}{(2\cos u)^3}+\cdots\right)$$

which expansion compared to this one:

$$\phi(\cos^2 u) = B_0 + B_1\frac{\cos u}{2\cos u} + B_2\frac{\cos 2u}{(2\cos u)^2} + B_3\frac{\cos 3u}{(2\cos u)^3} + \cdots$$

gives

66. $B_0 = A_0 + \frac{1}{2}A_1 + \frac{1\cdot 3}{2\cdot 4}A_2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}A_3 + \cdots,$

67. $B_1 = \frac{1}{2}A_1 + \frac{1\cdot 3\cdot 4}{2\cdot 4\cdot 3}A_2 + \frac{1\cdot 3\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 4}A_3 + \cdots,$

68. $B_2 = \frac{1}{2}A_1 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 5}{3\cdot 4}A_2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{6\cdot 7}{4\cdot 3}A_3 + \cdots,$

and it is easily seen that in general

69. $b_h = \frac{1}{2}A_1 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{h+3}{3}A_2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{(h+4)(h+5)}{4\cdot 5}A_3 + \cdots,$

the general term of which term is:

$$\frac{1\cdot 3\cdot 5\cdots(2p-1)\cdot(h+p+1)(h+p+2)\cdots(h+2p-1)}{2\cdot 4\cdot 6\cdots 2p\cdot(p+1)(p+2)\cdots(2p-1)}$$

which is easily transformed into this simpler form

$$\frac{(h+p+1)(h+p+2)\cdots(h+2p-1)}{2^{2p-1}\cdot 2\cdot 3\cdots(p-1)}A_p,$$

whence the series B_h takes on this form:

$$70. \quad B_h = \frac{A_1}{2} + \frac{h+3}{2^3 \cdot 1} A_2 + \frac{(h+4)(h+5)}{2^5 \cdot 1 \cdot 2} A_3 + \cdots$$

The sum of this series is deduced by means of theorem V. from this one

$$\psi(\cos^2 u) = A_1 + A_2 \cos^2 u + A_3 \cos^4 u + \cdots$$

for, if in the equations of this theorem V. one puts $\beta)h+2$, $\gamma = h+1$, changes A_0 into A_1, A_1 into A_2, A_2 into A_3 etc. and $\phi(\cos^2 u)$ into $\psi(\cos^2 u)$, it results:

$$A_1 + \frac{h+3}{2^2 \cdot 1} A_2 + \frac{(h+4)(h+5)}{2^4 \cdot 1 \cdot 2} A_3 + \cdots = \frac{2^{h+2}}{\pi} \int_0^{\frac{\pi}{2}} (\cos u)^{h+1} \cos(h+1) u \psi(\cos^2 u) du$$

therefore,

$$B_h = \frac{2^{h+1}}{\pi} \int_0^{\frac{\pi}{2}} (\cos u)^{h+1} \cos(h+1) u \psi(\cos^2 u) du;$$

but since $\psi(\cos^2 u) = \frac{\phi(\cos^2 u) - A_0}{\cos^2 u}$, we have

$$B_h = \frac{2^{h+1}}{\pi} \int_0^{\frac{\pi}{2}} (\cos u)^{h-1} \cos(h+1) u \phi(\cos^2 u) du$$

$$- \frac{A_0 2^{h+1}}{\pi} \int_0^{\frac{\pi}{2}} (\cos u)^{h-1} \cos(h+1) u du,$$

and since for each positive value of the number h

$$\int_0^{\frac{\pi}{2}} (\cos u)^{h-1} \cos(h+1) u du = 0$$

But it seems that the case $h = 0$ is to be excluded from this determination of the coefficients; for, the series B_0 differs from the remaining ones, since it contains the terms A_0; but by means of theorem IV. or V. one easily finds that this series is expressed by the integral

$$B_0 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \phi(\cos^2 u) du$$

whence it is clear that the first coefficient follows the same law as the other ones.

There is a remarkable relation among the coefficients of these series

$$72. \quad \phi(\cos^2 u) = B_= + B_1 \frac{\cos u}{2\cos u} + B_2 \frac{\cos 2u}{(2\cos u)^2} + B_3 \frac{\cos 3u}{(2\cos u)^3} + \cdots$$

and

$$73. \quad \phi(\cos^2 u) = C_0 + C_1 \cos 2u + C_2 \cos 4u + C_3 \cos 6u + \cdots,$$

for, if one substitutes this series for $\phi(\cos^2 u)$ in equation (71.), we find

$$74. \quad B_h = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} (2\cos u)^{h-1} \cos(h+1)u(C_0 + C_1 \cos 2u + C_2 \cos 4u + \cdots) du,$$

where the single terms of this form are to be integrated

$$\frac{4}{\pi} \int_0^{\frac{\pi}{2}} (2\cos u)^{h-1} \cos(h+1)u \cos(2ku) du,$$

which integral is divided into these two

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2\cos u)^{h-1} \cos(h+2k+1)u du + \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2\cos u)^{h-1} \cos(h-2k+1)u du,$$

which according to formula (30.) is expressed in terms of the function Π this way

$$\frac{\Pi(h-1)}{\Pi(h+k)\Pi(-k-1)} + \frac{\Pi(h-1)}{\Pi(h-k)\Pi(k-1)}$$

the first part always vanishes for each arbitrary integer number k, since then $\Pi(-k-1)=\infty$, the other part vanishes, if $h-k$ is a positive number, or if $k>h$, but if $k\leq h$, it goes over into

$$\frac{(h-1)(h-2)\cdots(h-k+1)}{1\cdot 2\cdots(k-1)};$$

therefore,

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}}(2\cos u)^{h-1}\cos(h+1)u\cos(2ku)du=\frac{(h-1)(h-2)\cdots(h-k+1)}{1\cdot 2\cdot 3\cdots(k-1)},$$

hence equation (74.) goes over into this one

$$75.\quad B_k=C_1+\frac{h-1}{1}C_2+\frac{h-1)(h-2)}{1\cdot 2}C_3+\cdots+C_h;$$

furthermore, since the first term of the series (73.) is expressed by the integral

$$C_0=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\phi(\cos^2 u)du$$

it follows

$$B_0=C_0,$$

therefore, the coefficients of the expansion (72.) can be found most easily from the coefficients of the series (73.),
Lignietz, in the month of May, 1836